Optimal Impulsive Time-Fixed Direct-Ascent Interception

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Minimum-fuel, impulsive, time-fixed solutions are obtained for the problem of direct-ascent interception of a target in circular orbit from a launch point on the surface of a planet or moon. Both a nonrotating and rotating planet are considered, as well as coplanar (launch point in target orbital plane) and noncoplanar cases. In the rotating planet model, the target orbit plane is assumed to be equatorial. Atmospheric effects are neglected, and the planet surface constraint is included. Primer vector theory is used to obtain the optimal solutions, which often include an initial coast interval prior to the first thrust impulse. Optimal intercept solutions obtained include both one- and two-impulse posigrade and retrograde trajectories. Solutions are obtained for a range of fixed transfer times, target orbit radii and inclinations, and initial target phase angles relative to the launch point.

I. Introduction

OPTIMAL time-fixed impulsive intercept trajectories from a point at rest on the surface of a planet or moon to a target in circular orbit are obtained. Some of the results in this study are in Heckathorn[1] and form a companion analysis to the related problem of optimal direct-ascent rendezvous treated previously by Gross and Prussing.[2] Previous optimal intercept studies in the literature deal primarily with time-open disorbit problems.[3]

Applications of the results obtained include interception or reconnaissance of orbiting satellites and deployment of satellites at specified locations that are intercepted by the deploying vehicle. The idealized results obtained assuming impulsive thrust and absence of atmospheric effects can be applied to more realistic cases by adding gravity and drag losses to the velocity requirements obtained to account for finite burn times and atmospheric drag. In addition, the case of airborne launch rather than launch from the surface can be obtained by interpreting the launch conditions differently, e.g., the contact force of the planet prior to launch becomes the lift on the aircraft. The optimal time-fixed results obtained can be used to perform time vs fuel trade-offs for missions that have operational time constraints.

II. Necessary Conditions for an Optimal Interception

The basic necessary conditions for an optimal impulsive trajectory and their applications are well documented elsewhere, for example, by Lawden,[4] Lion and Handelsman,[5] and Prussing and Chiu.[6] These conditions are conveniently expressed in terms of the primer vector, which is the adjoint to the velocity vector of the vehicle. The primer $\boldsymbol{p}$ satisfies the linear differential equation

$$\ddot{\boldsymbol{p}} = G(\boldsymbol{r})\boldsymbol{p} \tag{1}$$

where $G(\boldsymbol{r})$ is the gravity gradient matrix for the gravitational field. For an inverse-square gravitational force considered in this study, Eq. (1) becomes

$$\ddot{\boldsymbol{p}} = \mu(3\boldsymbol{r}\boldsymbol{r}^T - r^2 I)\boldsymbol{p}/r^5 \tag{2}$$

Convenient forms of the solution to Eq. (2) are given by Glandorf[7] and by Gravier, Marchal, and Culp.[8]

The necessary conditions for an optimal impulsive trajectory, first derived by Lawden[4] are:

1) The primer vector satisfies Eq. (2) and must be continuous with continuous first derivative everywhere.

2) The magnitude $p \leq 1$ during the transfer, with the impulses occurring at those instants for which $p = 1$.

3) At an impulse time, the primer vector is a unit vector in the optimal thrust direction.

4) As a consequence of condition 2, $\dot{p} = \dot{\boldsymbol{p}}^T\boldsymbol{p} = 0$ at all interior impulses (not at the initial or final time).

As an extension, Lion and Handelsman[5] demonstrated that the primer vector evaluated along a nonoptimal trajectory provides information on how the cost can be decreased by including additional impulses or terminal coasting periods.

The major difference between the present study and the orbit-to-orbit rendezvous problem treated in Ref. 6 are: 1) for an optimal interception, the final velocity is unspecified; and 2) at the initial time, the vehicle is not orbiting the central body but is at rest on the surface of it. As a consequence of point 1, the transversality condition prescribes the primer vector at the final time to be the zero vector (see Sec. V). As a consequence of point 2, during an initial coast period prior to the first impulse, the only change in initial vehicle position is due to planet rotation. Prior to launch, the equations of motion include an additional force besides gravity acting on the vehicle, namely, the contact force on the vehicle from the planet surface. In addition, the planetary surface introduces a constraint on the position of the vehicle: the magnitude of the position vector at any point on the trajectory can never be less than the planet radius.

The criterion for including an additional impulse and the procedure for adjusting the time and location of a midcourse impulse are based on those developed in Ref. 5 and applied in Ref. 6. This procedure is discussed in Sec. IV.

The criterion for including an initial coast period prior to the launch impulse, however, differs significantly from the orbit-to-orbit criterion of Ref. 5 and is very important to this study. Figure 1 depicts the change in launch geometry (caused by planet rotation, if any) resulting from an initial coast of duration $dt_0 > 0$. For both trajectories Γ and $\hat{\Gamma}$, the intercept occurs at position $\boldsymbol{r}(t_f)$ and time t_f. On trajectory Γ, the launch impulse

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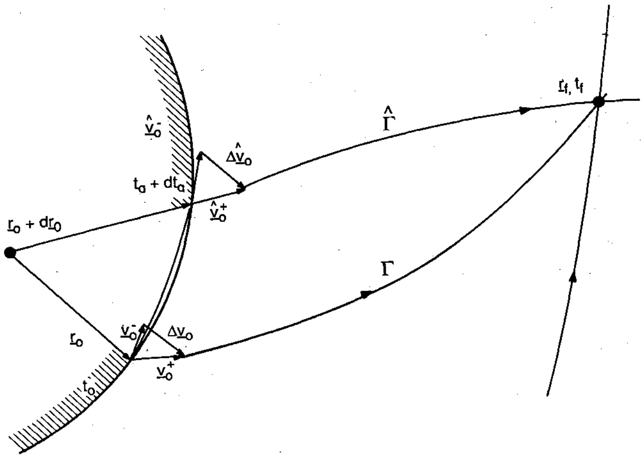

Fig. 1 Comparison trajectories for an initial coast.

occurs at time t_0 whereas, on the delayed trajectory $\hat{\Gamma}$, the launch impulse occurs at the later time $t_0 + \mathrm{d}t_0$. The velocities prior to the launch are $\boldsymbol{v}_0^-$ and $\hat{\boldsymbol{v}}_0^-$ and the launch position changes during the coast by an amount $\mathrm{d}\boldsymbol{r}_0 = \boldsymbol{v}_0^- \mathrm{d}t_0$, for which

$$\boldsymbol{v}_0^- = \boldsymbol{\omega}_p \times \boldsymbol{r}_0 \tag{3}$$

when $\boldsymbol{\omega}_p$ is the (constant) planet rotational angular velocity. The rate of change of the velocity prior to launch is given by

$$\dot{\boldsymbol{v}}_0^- = \boldsymbol{\omega}_p \times \boldsymbol{v}_0^- = \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_p \times \boldsymbol{r}_0) = (\boldsymbol{\omega}_p^T \boldsymbol{r}_0)\boldsymbol{\omega}_p - \omega_p^2 \boldsymbol{r}_0 \tag{4}$$

The required velocities after the impulses $\boldsymbol{v}_0^+$ and $\hat{\boldsymbol{v}}_0^+$, which occur as a result of the differences in initial position and flight time owing to the initial coast, are determined by solving Lambert's problem. The new algorithm of Battin and Vaughan[9] was used for this and proved to be very efficient. The required velocity changes $\Delta\boldsymbol{v}_0$ and $\Delta\hat{\boldsymbol{v}}_0$ are then calculated as the vector differences shown in Fig. 1.

To satisfy the necessary conditions for an optimal trajectory, the primer vector must be a unit vector in the optimal thrust direction at an impulse time. This determines the initial condition on the primer vector as

$$\boldsymbol{p}(t_0) = \Delta\boldsymbol{v}_0 / \Delta v_0 \tag{5}$$

where Δv_0 is the initial velocity change. As mentioned previously, the final condition for an optimal interception is

$$\boldsymbol{p}(t_f) = \boldsymbol{0} \tag{6}$$

The solution for the primer vector along the trajectory is then determined by Eq. (2) and the terminal conditions [Eqs. (5) and (6)].

III. Determination of an Optimal Coast

If the primer vector does not satisfy the necessary conditions for an optimal trajectory, an iterative procedure can be established based on the gradient relating a differential change in the cost $\mathrm{d}J$ due to an initial coast of duration $\mathrm{d}t_0$. Whenever $\mathrm{d}J < 0$ for $\mathrm{d}t_0 > 0$, a small initial coast will decrease the cost.

It is convenient to use the notation d( ) to represent a noncontemporaneous variation in a variable (including the effect of the time difference $\mathrm{d}t_0$), as contrasted with $\delta($ $)$, which denotes the contemporaneous variation ($\delta t \equiv 0$). The relationship between these two variations is

$$\mathrm{d}(\underline{\ }) = \delta(\underline{\ }) + (\dot{\underline{\ }})\,\mathrm{d}t_0 \tag{7}$$

The difference in cost between trajectories Γ and $\hat{\Gamma}$ in Fig. 1 is

$$\mathrm{d}J = |\hat{\boldsymbol{v}}_0^+ - \hat{\boldsymbol{v}}_0^-| - |\boldsymbol{v}_0^+ - \boldsymbol{v}_0^-| \tag{8}$$

which can be written in terms of

$$\mathrm{d}\boldsymbol{v}_0^+ = \hat{\boldsymbol{v}}_0^+ - \boldsymbol{v}_0^+$$
$$\mathrm{d}\boldsymbol{v}_0^- = \hat{\boldsymbol{v}}_0^- - \boldsymbol{v}_0^- \tag{9}$$

as

$$\mathrm{d}J = |\boldsymbol{v}_0^+ - \boldsymbol{v}_0^- + \mathrm{d}\boldsymbol{v}_0^+ - \mathrm{d}\boldsymbol{v}_0^-| - |\boldsymbol{v}_0^+ - \boldsymbol{v}_0^-|$$
$$= |\Delta\boldsymbol{v}_0 + \Delta\,\mathrm{d}\boldsymbol{v}_0| - |\Delta\boldsymbol{v}_0| \tag{10}$$

To first order, this can be expressed as

$$\mathrm{d}J = \Delta\boldsymbol{v}_0^T \Delta\,\mathrm{d}\boldsymbol{v}_0 / \Delta v_0 = \boldsymbol{p}_0^T \Delta\,\mathrm{d}\boldsymbol{v}_0 \tag{11}$$

where the boundary condition [Eq. (5)] on the initial primer vector has been used. Using Eq. (7) to relate the variations

$$\mathrm{d}\boldsymbol{v}_0^+ = \delta\boldsymbol{v}_0^+ + \dot{\boldsymbol{v}}_0^+\,\mathrm{d}t_0$$
$$\mathrm{d}\boldsymbol{v}_0^- = \dot{\boldsymbol{v}}_0^-\,\mathrm{d}t_0$$
$$\Delta\,\mathrm{d}\boldsymbol{v}_0 = \delta\boldsymbol{v}_0^+ + (\dot{\boldsymbol{v}}_0^+ - \dot{\boldsymbol{v}}_0^-)\,\mathrm{d}t_0 \tag{12}$$

where $\delta\boldsymbol{v}_0^- = \boldsymbol{0}$ has been used (the only change in $\boldsymbol{v}_0^-$ is due to the initial coast).

The important difference between Eq. (12) and an analogous result obtained in Ref. 5 for the orbit-to-orbit case is that, in the present application, the acceleration $\dot{\boldsymbol{v}}$ is not continuous across the launch impulse. The value of $\dot{\boldsymbol{v}}_0^- = \boldsymbol{g}_0 + \boldsymbol{N}$, for an initial coast on the planet surface, where $\boldsymbol{g}_0$ is the gravitational acceleration and $\boldsymbol{N}$ the contact force per unit mass on the vehicle from the planet surface. An expression for this acceleration is given in Eq. (4) and is equal to zero for a nonrotating planet ($\boldsymbol{N} = -\boldsymbol{g}_0$). Using this and the fact that $\dot{\boldsymbol{v}}_0^+ = \boldsymbol{g}_0$ in Eq. (12) allows Eq. (11) to be rewritten as

$$\mathrm{d}J = \boldsymbol{p}_0^T(\delta\boldsymbol{v}_0^+ - \boldsymbol{N}\,\mathrm{d}t_0) \tag{13}$$

An explicit expression for the contact force per unit mass $\boldsymbol{N}$ is obtained using Eq. (4) $\boldsymbol{g}_0 = -\mu\boldsymbol{r}_0/r_0^3$ as

$$\boldsymbol{N} = (\omega_0^2 - \omega_p^2)\boldsymbol{r}_0 + (\boldsymbol{\omega}_p^T\boldsymbol{r}_0)\boldsymbol{\omega}_p \tag{14}$$

where $\omega_0^2 = \mu/r_0^3$ is the square of the orbital angular velocity (mean motion) in circular orbit at the planet surface. As one would expect, at a point on the equator ($\boldsymbol{\omega}_p^T\boldsymbol{r}_0 = 0$), $\boldsymbol{N}$ would vanish if the planet were rotating so fast that the rotation period was equal to circular orbit period at the surface ($\omega_p = \omega_0$). In the typical case, however, $\omega_p/\omega_0 \ll 1$; for the Earth, it is approximately 6×10^{-2} and for the moon, it is approximately 3×10^{-3}.

Equation (13) for the differential cost $\mathrm{d}J$ can be further simplified using a variational constant that is based on the definition of the adjoint system and is derived in Refs. 5 and 6:

$$\boldsymbol{p}^T\delta\boldsymbol{v} - \dot{\boldsymbol{p}}^T\delta\boldsymbol{r} = \text{const} \tag{15}$$

For an optimal interception the value of the constant is zero because $\boldsymbol{p}(t_f) = \delta\boldsymbol{r}(t_f) = \boldsymbol{0}$. Thus, $\boldsymbol{p}_0^T\delta\boldsymbol{v}_0^+ = \dot{\boldsymbol{p}}_0^T\delta\boldsymbol{r}_0$ and on the transfer orbit

$$\delta\boldsymbol{r}_0 = \mathrm{d}\boldsymbol{r}_0 - \boldsymbol{v}_0^+\,\mathrm{d}t_0 \tag{16}$$

where $\mathrm{d}\boldsymbol{r}_0 = \boldsymbol{v}_0^-\,\mathrm{d}t_0$ owing to the planet rotation. Thus, $\delta\boldsymbol{r}_0 = -\Delta\boldsymbol{v}_0\,\mathrm{d}t_0$ and the final form of Eq. (13) can be written as

$$\mathrm{d}J = -\boldsymbol{p}_0^T[\Delta V_0\dot{\boldsymbol{p}}_0 + (\omega_0^2 - \omega_p^2)\boldsymbol{r}_0 + (\boldsymbol{\omega}_p^T\boldsymbol{r}_0)\boldsymbol{\omega}_p]\,\mathrm{d}t_0 \tag{17}$$

The first term in Eq. (17) is the result obtained in Ref. 5 for an initial coast on an initial orbit. In that case, an initial coast decreases the cost ($\mathrm{d}J < 0$) if $\boldsymbol{p}_0^T\dot{\boldsymbol{p}}_0 > 0$, i.e., if the primer magnitude exceeds unity after the first impulse. The additional terms in Eq. (17) are due to the fact that the launch point is at rest on the planet surface rather than in orbit. If the entire coefficient of dt_0 in Eq. (17) is negative, the cost can be decreased by a small initial coast. The condition for an optimal initial coast is a zero value for the coefficient.

Two further comments can be made concerning Eq. (17). First, it is equivalent to the more familiar transversality condition[5]

$$dJ = \dot{p}_0^T \, dr_0 - p_0^T \, dv_0 - H_0 \, dt_0 \tag{18}$$

where H is the Hamiltonian function

$$H = p^T g - \dot{p}^T v \tag{19}$$

Second, in order to satisfy the planet surface constraint $r_0^T v_0^+ \geq 0$. However, $r_0^T v_0^+ = r_0^T \Delta V_0$, even in the rotating planet case, because v_0^- is normal to r_0 [Eq. (3)]. Thus, for an optimal solution, in which $\Delta V_0 = \Delta v_0 p_0$, satisfaction of the planet surface constraints requires that $r_0^T p_0 \geq 0$. From Eq. (17), it is evident that even though $p_0^T \dot{p}_0 = \dot{p}_0 < 0$ (the primer magnitude decreases after the initial impulse), an initial coast may still decrease the cost, unlike the orbit-to-orbit case. Conversely, if the primer magnitude increases after the initial impulse and the surface constraint is satisfied, an initial coast will decrease the cost.

An interesting application of Eq. (17) occurs in the case of a nonrotating planet ($\omega_p = 0$). In this case, Eq. (17) reduces to

$$dJ = -p_0^T(\Delta v_0 \dot{p}_0 + \omega_0^2 r_0) \, dt_0 = H_0^+ \, dt_0 \tag{20}$$

A special case that satisfies the condition $dJ = 0$ for an optimal initial coast occurs if the term in parentheses in Eq. (20) is equal to the zero vector. Because $v_0^- = \mathbf{0}$ if there is no planet rotation, $\Delta v_0 = v_0^+$ on the trajectory and the term in parentheses will be zero if

$$\dot{p}_0 = -\omega_0^2 r_0 / v_0 = g_0 / v_0 = \dot{v}_0 / v_0 \tag{21}$$

This represents the initial value for the primer derivative $\dot{p}(t) = \dot{v}(t)/v_0$, which corresponds to a primer vector $p(t) = v(t)/v_0$. This is a valid solution to the primer vector Eq. (1). (Also, see Ref. 7 for the case $\alpha = \gamma = 0$, $\beta = 1/v_0$ and Ref. 8 for the case $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_1 = 1/v_0$.) Because it is initially a unit vector in the Δv_0 direction, this primer vector also satisfies the necessary conditions for an optimal intercept if $v(t_F) = \mathbf{0}$.

The trajectory in this case is the rectilinear ellipse, which achieves maximum altitude (zero velocity) at the target orbit. This rectilinear ellipse is the optimal initial coast trajectory for the case of a nonrotating planet and a launch point in the target orbit plane. As shown in Sec. V, this same rectilinear ellipse also satisfies the conditions for the time-open optimal solution. Based on these facts and the discussion of minimum energy ellipses in Sec. VI, it is evident that this rectilinear ellipse is the global time-open optimal intercept trajectory for the coplanar nonrotating case.

IV. Constrained Optimal Trajectories

For some final target locations and a sufficiently small transfer time, the optimal one-impulse trajectory passes through the planet and violates the planet surface constraint. A constrained optimal solution is then sought that will require at least one midcourse impulse in addition to the launch impulse in order to avoid violating the planet surface constraint. A particularly simple two-impulse trajectory was found to be the constrained optimal solution in all cases except the extremely high-fuel-cost trajectories, which occur as the specified transfer time tends to zero. This two-impulse constrained optimal trajectory consists of two segments: 1) a circular orbit along the planet surface, followed by 2) a conic orbit that transfers the vehicle from the constraint surface to the target in the time remaining. The analysis that follows assumes an equatorial launch point but can be readily generalized. The basic relationship used to determine the optimal location r_m and time t_m of a midcourse impulse is given by Lion and Handelsman[5] as

$$dJ = (\dot{p}_m^+ - \dot{p}_m^-)^T \, dr_m + (H_m^+ - H_m^-) \, dt_m \tag{22}$$

where the subscript m refers to the midcourse impulse time and the superscripts + and − signify values immediately after and before the midcourse impulse, respectively.

The gradients of the cost with respect to r_m and t_m given in Eq. (22) determine how the midcourse impulse position and time should be iteratively adjusted to decrease the cost ($dJ < 0$). However, the values of r_m and t_m are *not* independent on the circular orbit segment. They are related by

$$dr_m = v_m^- \, dt_m \tag{23}$$

where v_m^- is circular orbit velocity at the midcourse impulse point. Equation (22) then becomes

$$dJ = [(\dot{p}_m^+ - \dot{p}_m^-)^T v_m^- + H_m^+ - H_m^-] \, dt_m \tag{24}$$

We then iterate on the scalar variable t_m using the gradient given in Eq. (24) to lower the cost subject to two additional conditions: 1) $r_m^T v_m^+ \geq 0$ and 2) $(\dot{p}_m^+ - \dot{p}_m^-)^T r_m \geq 0$. Condition 1 must be satisfied in order that the second conic segment not violate the planet surface constraint. If condition 2 is satisfied, the optimal midcourse impulse is indeed on the planet surface and not at a larger radius; moving the impulse point off the surface will cause a first-order increase in the cost. In the typical case, condition 1 terminates the iteration on t_m, resulting in the second conic segment being tangent to the planet surface.

Along the first (circular) segment of the constrained optimal solution, the primer vector is simply a unit vector in the direction of the velocity vector for both nonrotating and rotating planets. This represents a singular arc along which the primer magnitude is identically unity. Because of this fact, the Hamiltonian of Eq. (19) is identically zero and, by Eq. (20), an initial coast will never decrease the cost to first order on the constrained optimal solution.

V. Time-Open Optimal Conditions

The transversality condition at the final time must be satisfied for both optimal time-fixed and time-open intercept trajectories. The expression for the variation in cost due to variations in the terminal conditions at time t_F is similar to Eq. (18):

$$dJ = p_F^T \, dv_F - \dot{p}_F^T \, dr_F - H_f \, dt_F \tag{25}$$

For an interception, it is necessary in order that $dJ = 0$ that $p_F = \mathbf{0}$ because dv_F is arbitrary. For the optimal time-fixed case, $dr_F = \mathbf{0}$ and $dt_F = 0$, so that no other conditions are required.

For the optimal time-open interception, however, an additional condition must be satisfied to determine the best value of t_F and, hence, r_F. In this case, $dt_F \neq 0$, and dr_F is due to the motion of the target:

$$dr_F = v_F^{\#} \, dt_F \tag{26}$$

where $v_F^{\#}$ is the velocity of the target at time t_F. Using Eqs. (19) and (26), and the fact that $p_F = \mathbf{0}$, Eq. (25) yields

$$dJ = \dot{p}_F^T(v_F - v_F^{\#}) \, dt_F \tag{27}$$

Thus, for an optimal time-open interception, at the final time, the primer derivative must be orthogonal to the *relative* velocity (of the vehicle relative to the target). This condition is illustrated in the zero-gravity solution given in the Appendix.

Note that it is not required that H_F be zero on the time-open optimal solution. However, because H is constant along the trajectory, if the initial value H_0 is zero as a result of an optimal coast on a nonrotating planet, $H_F = -\dot{p}_F^T v_F = 0$ which, combined with Eq. (27), implies that the final primer derivative is orthogonal to the plane formed by V_F and $v_F^\#$.

As mentioned in Sec. III the rectilinear ellipse having maximum altitude at the target orbit satisfies the conditions for an optimal time-open solution. Equation (27) is satisfied because the vehicle velocity v_F is zero and the target velocity $v_F^\#$ is orthogonal to the primer derivative. Thus, this rectilinear ellipse is the time-open optimal solution for a nonrotating planet and applies when the launch point lies in the target orbit plane. The Hamiltonian function on the trajectory is equal to zero.

VI. Optimal Intercept Trajectories

The geometry of the intercept problem is conveniently described in terms of a planet-centered, inertial Cartesian coordinate frame. The target orbit lies in the XY plane, which is the equatorial plane of the planet. The launch point at the initial time is in the XZ plane at a latitude ϕ. Note that, for a nonrotating planet, this formulation is completely general because the equatorial plane is arbitrary. The target is in circular orbit of radius R at an initial location described by the lead angle β, measured from the positive x axis in the direction of target motion.

The units used in this analysis are normalized so that the unit of length is the planet radius. The unit of time is chosen so that the period of a circular orbit at the surface of the planet (unit radius) is 2π time units. In terms of these normalized units, both circular orbit speed at the planet surface and the gravitational constant μ of the planet have unit magnitudes.

Because the final condition is an interception as opposed to a rendezvous, both posigrade and retrograde trajectories from the launch point to the target can be optimal, with posigrade implying motion in the same sense as the target. In the rotating planet case, the target orbit itself is posigrade with respect to planet rotation. Whether the posigrade or retrograde trajectory is optimal depends on the specified transfer time, the initial target radius R and initial lead angle β, and the constraint that the trajectory must remain outside the planet.

Nonrotating Planet

In the case of a nonrotating planet, the optimal initial coast one-impulse time-fixed and time-open intercept trajectories can be deduced using the concept of a minimum-energy ellipse. Whether these solutions are optimal compared to multiple impulse trajectories is determined by evaluating the primer vector along the trajectory and determining whether the necessary conditions are satisfied or whether an additional impulse or an initial coast is optimal. Even if the necessary conditions are satisfied, global optimality is not guaranteed, but it can be inferred if the solutions approach a predicted time-open global solution as the transfer time is varied.

For a nonrotating planet, the velocity change owing to the launch impulse and the initial velocity on the trajectory are identical because the velocity prior to the impulse is zero. Thus, the minimum Δv case occurs when the initial velocity at the launch point on the planet surface is a minimum, i.e., the trajectory is a minimum-energy ellipse[10,11] to the target. A basic geometrical property of the minimum-energy ellipse is that the value of its semimajor axis a_m is equal to one-fourth the perimeter of the "space triangle" formed by the initial radius vector r_0, the final radius vector r_f, and the chord connecting the terminal points $c = r_f - r_0$. Associated with the minimum-energy ellipse is a unique transfer time t_m, which is easily calculated using the equations for Lambert's problem.[10,11]

The optimal one-impulse time-fixed solution is the minimum-energy ellipse obtained using the geometrical property described earlier. The initial radius r_0 is fixed, but the orientation of the final radius r_f is a function of the transfer time t_f for a given initial phase angle β. The time-open optimal value of t_f is that for which the perimeter of the space triangle is the smallest possible. For the coplanar case, in which the launch point lies in the target orbit plane, this occurs when the initial and final radii are parallel, i.e., a rectilinear orbit. The time-open optimal value of t_f^* is then simply the time required for the target to travel from its initial position to the point overhead of the launch point:

$$t_F^* = (1 - \beta/2\pi)P^* \tag{28}$$

The value of the semimajor axis a_m in this case is then $R/2$, where R is the target orbit radius, which implies that the maximum radius is $2a_m = R$, at which point the final velocity is zero.

The optimal time-open coplanar trajectory for a given R and β comprises an initial coast until the rectilinear trajectory is available that will intercept the target directly overhead of the launch point with transfer time t_m^* and zero velocity relative to the center of the planet. (The vehicle velocity relative to the target will not be zero but equal to the target orbital velocity). The Δv^* required depends only on the target orbit radius and in the chosen units is given by

$$\Delta v^* = [2(R-1)/R]^{\frac{1}{2}} \tag{29}$$

For the noncoplanar case, the analogous result is that the time-open trajectory intercepts the target at the same longitude as the launch point (but not the same latitude because the target orbit is equatorial). This geometry minimizes the perimeter of the space triangle, resulting in a minimum-energy trajectory that is not rectilinear but lies in the XZ plane and has a transfer time t_m^*.

Based on the time-open optimal solutions, the time-fixed optimal solutions can be determined. Once t_F is specified, the final target position for a given β is known, and a value t_m can be calculated. If the specified $t_F \geq t_m$, the optimal solution is composed of a coast of duration $t_F - t_m$, followed by the minimum-energy trajectory of duration t_m. For the case in which the specified $t_F < t_m$, a minimum-energy trajectory is not available. We must then examine no-coast posigrade and retrograde orbits and determine whether the planet surface constraint is violated and whether an initial coast is optimal, according to the condition of Eq. (17). If an initial coast is indicated, a bisection iteration method is used to determine the initial coast time that yields $dJ = 0$.

Numerical results for coplanar nonrotating cases are presented in Figs. 2 and 3 for values of target orbit radius $R = 1.1$ and 6.61. The latter represents a 24-h target orbit period for the Earth.

Figure 2 for $R = 1.1$ illustrates the basic structure of the optimal solutions. Four values of initial target lead angle are

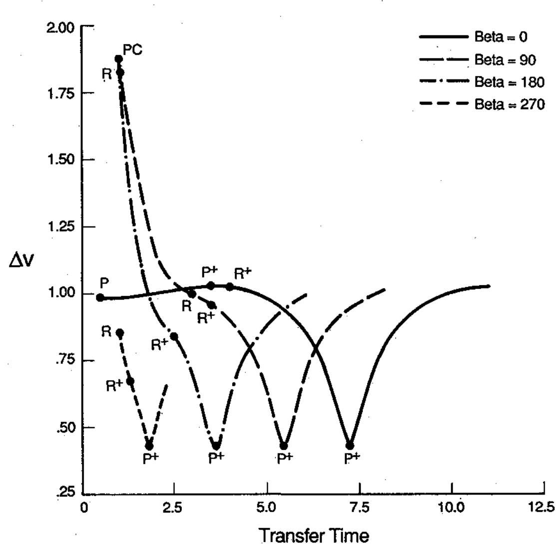

Fig. 2 Optimal ΔV vs transfer time for $R = 1.1$.

shown; $\beta = 0$, 90, 180, and 270 deg. The symbol P indicates a posigrade trajectory, R retrograde, and C a violation of the planet surface constraint. A superscript + indicates that an initial coast period is optimal. A symbol applies to all data points to the right of the labeled point unless otherwise indicated.

The minimum point on each curve in Fig. 2 occurs at the time-open optimal (rectilinear) solution time t_F^* with Δv^* given by Eq. (29) equal to 0.426 and values of t_F^* given by Eq. (28) as multiples of $P^\#/4$, where $P^\#$ is the target orbit period 7.249 time units. As mentioned previously, for specified transfer times greater than t_F^* on each curve, the optimal solution is P^+, representing a coast until the minimum-energy trajectory is available. For values $t_F < t_F^*$, a retrograde trajectory with or without coast is typically optimal except for the short transfer times for $\beta = 0$ where posigrade is again optimal.

The basic structure of Fig. 3 for $R = 6.61$ corresponding to $P^\# = 107.1$ is similar to the $R = 1.1$ case but with correspondingly higher Δv values and transfer times t_F^*. Note also that, for larger values of R, the variation in Δv is less. In Fig. 3, except for very short times, the variation is less than 2%.

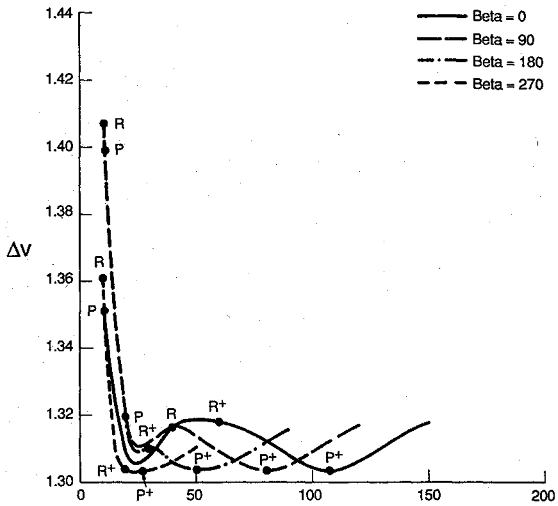

Fig. 3 Optimal ΔV vs transfer time for $R = 6.61$.

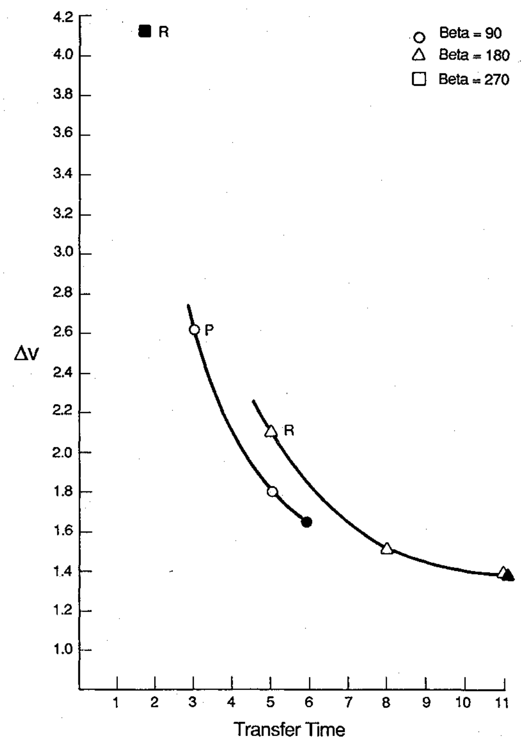

Fig. 4 Optimal ΔV vs transfer time for $R = 6.61$ with planet surface constraint.

Figure 4 displays the constrained optimal two-impulse solution required to satisfy the planet surface constraint for smaller transfer times than are shown in Fig. 3 for $R = 6.61$. The filled-in symbol identifying the value of β designates the minimum time for which a one-impulse solution satisfies the surface constraint, discussed in Sec. IV. For $R = 4.17$, the two-impulse solution is required if $t_f < 6.03$ for $\beta = 90$ deg and if $t_f < 6.55$ for $\beta = 180$ deg. In the $R = 6.61$ case, the two-impulse solution is required if $t_f < 5.92$ for $\beta = 90$ deg and if $t_f < 11.2$ for $\beta = 180$ deg. Note that a transfer time of 11.2 is significantly large, being almost two circular orbit periods at the surface of the planet. For a target orbiting the Earth with an initial target angle of $\beta = 180$ deg, this means that all transfer times less than 2.5 h require the use of the two-impulse constrained optimal solution.

The case of $\beta = 0$ is not shown in Fig. 4 because the one-impulse solution satisfies the constraint for arbitrarily small transfer times (the vehicle is launched vertically upward). The minimum one-impulse time for $\beta = 270$ deg is shown on the figure, but it is so small that solutions are not obtained for shorter transfer times. For both $\beta = 180$ and 270 deg, the optimal solution is retrograde as indicated. In the case of a rotating planet, because the launch point is assumed equatorial in Fig. 4, the ΔV values shown are modified by subtracting (for a posigrade trajectory) or adding (retrograde) the velocity of the

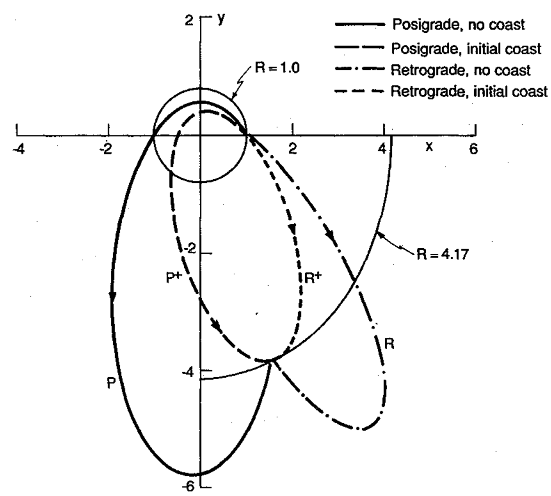

Fig. 5 Example trajectories for $R = 4.17$, $\beta = 90$, $t_f = 30$.

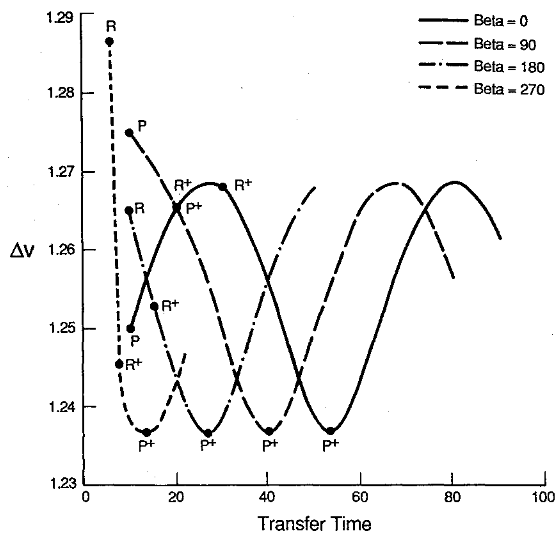

Fig. 6 Optimal ΔV vs transfer time, noncoplanar, $R = 4.17$.

launch point $\omega_p r_0$. For Earth, this has a value of approximately 0.06 in the normalized velocity units used in the figures.

Figure 5 displays four sample trajectories for $R = 4.17$, $\beta = 90$ deg, and $t_F = 30$. In the figure, posigrade motion is counterclockwise. For the case shown, both the P trajectory and P^+ posigrade minimum-energy transfer violate the planet surface constraint. Both the P and R trajectories travel outside the target orbit to a radius of approximately six before falling back to intercept the target. The optimal trajectory is an R^+ with a coast of 19.49 time units followed by a minimum-energy retrograde elliptical trajectory to the target.

Figure 6 displays a noncoplanar case for $R = 4.17$, in which the launch point is at 28 deg latitude. The basic structure is similar to the coplanar case with the Δv costs being slightly higher because of the out-of-target-orbit plane motion. The Δv^* for the noncoplanar case is 1.237 compared with 1.233 for the coplanar case.

Rotating Planet

The rotating planet model is more complicated because the velocity change and the velocity after the impulse are not equal, as the velocity prior to the impulse is nonzero. Thus minimum Δv and minimum-energy trajectories are not equivalent, and the time-open optimal solution is not easily deduced as in the nonrotating case. For a specified value of t_F, we must iterate on Eq. (17) to determine the optimal initial coast, for which $dJ = 0$.

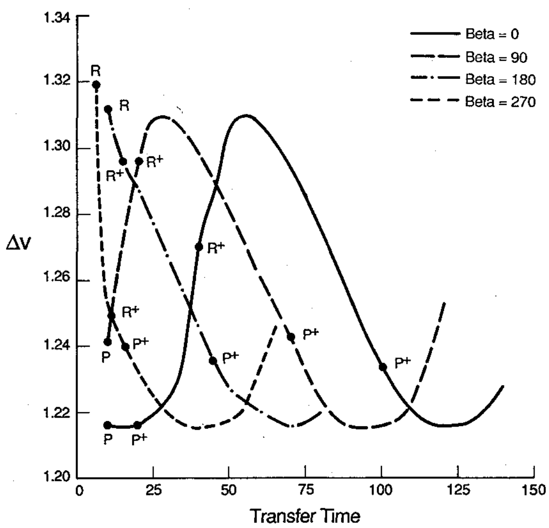

Fig. 7 Optimal ΔV vs transfer time, noncoplanar and rotating planet, $R = 4.17$.

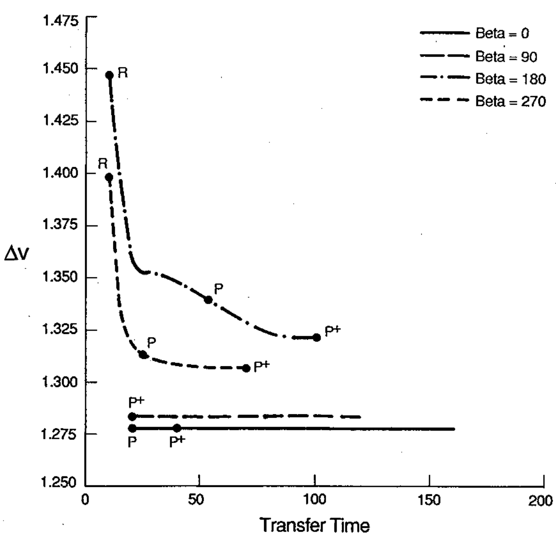

Fig. 8 Optimal ΔV vs transfer time, synchronous rotation, $R = 6.61$.

Figure 7 shows the results analogous to those of Fig. 6 but for which the nondimensional planet rotation rate is $\omega_p = 0.0589$, which is approximately the value for Earth. Even for this relatively low planet rotation rate, the contrasts are striking.

One major difference is the values of the time-open optimal transfer times t_F^* for the various values of β. Because the launch point is rotating, the period of the relative motion of the target with respect to the launch point is the synodic period

$$S = \frac{2\pi}{\omega^* - \omega_p} > \frac{2\pi}{\omega^*} \tag{30}$$

where ω^* is the mean motion of the target orbit. In the nonrotating planet model, the period of the target motion with respect to the launch point is simply the target orbit $P^* = 2\pi/\omega^*$. For the case illustrated in Fig. 7, $S = 107.3$ compared with $P^* = 53.54$, a factor of approximately 2. Thus, the times for data points on the various curves in Fig. 7 are approximately twice the values in Fig. 6. The values of Δv^* are different in the rotating model because the posigrade trajectories require less Δv as a result of the easterly velocity of the launch point itself and retrograde trajectories requiring higher Δv.

The time-open optimal times t_F^* are greater than twice the value for the nonrotating case because, in the coplanar case, the optimal intercept does not occur overhead of the launch point but slightly down-orbit. This trend is also evident in the noncoplanar results. In Fig. 6, for $\beta = 0$, $t_F^* = P^* = 53.54$, whereas t_F^* in Fig. 7 for this case is clearly greater than $S = 107.3$. Note also that, in Fig. 7, the values of t_f^* for the various values of β differ by multiples of $S/4$.

An even more striking characteristic of the rotating planet model is shown in Fig. 8, which depicts the case of a target orbit that is synchronous with the planet rotation. In this case, the geometry of the target relative to the launch point is invariant, and the Δv curve for each value of β converges to a minimum value, representing a posigrade trajectory. For all transfer times greater than the minimizing value, the optimal solution is P^+, where the coast time represents a waiting period until the correct amount of time remains to make the absolute minimum Δv transfer. Unlike Fig. 8, all other Δv vs t_F plots are periodic according to the synodic period between the target and the launch point. For a synchronous target orbit, the synodic period is infinite as shown by Eq. (30). Also unique to the synchronous case of Fig. 8 is the fact that initial coasts are nonoptimal for transfer times less than the minimizing

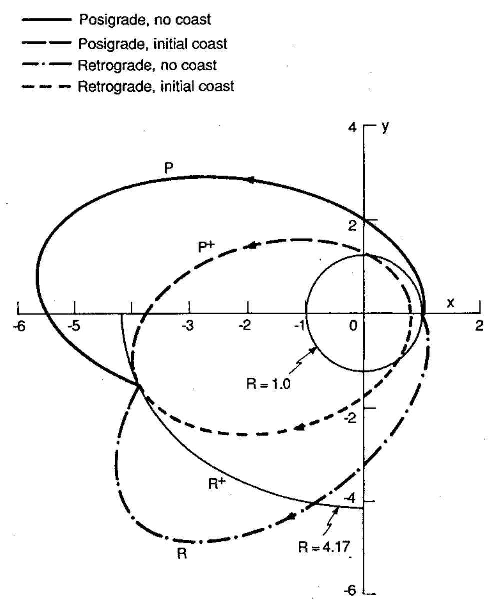

Fig. 9 Example trajectories for rotating planet, $R = 4.17$.

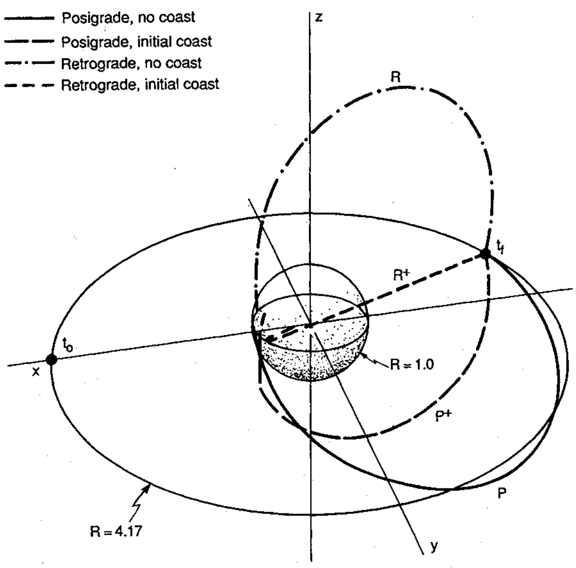

Fig. 10 Example noncoplanar trajectories, $R = 4.17$.

value, the reason being that there is no change in relative geometry during an initial coast.

Figures 9 and 10 display sample trajectories for the rotating planet model. Figure 9 illustrates the coplanar case for $R = 4.17$, $\beta = 0$ deg, and $t_f = 30$. Posigrade motion is counterclockwise. The R and P trajectories depart the planet at a point on the x axis, but the best R^+ and P^+ trajectories depart from (different) rotated points in the first quadrant of the xy plane. The optimal solution in this figure is P^+. The best R^+ trajectory violates the planet surface constraint, and the P and R solutions travel radially past the target altitude and, in doing so, incur unnecessary Δv costs.

Figure 10 illustrates sample trajectories for the noncoplanar, rotating planet model for $R = 4.17$, $\beta = 0$, $t_f = 30$, and launch latitude $\phi = 28$ deg. This case corresponds to a data point in Fig. 7 for which the P^+ trajectory is optimal. Similarly to the case of Fig. 9, the R and P trajectories travel radially outside the target, and the best R^+ trajectory violates the planet surface constraint.

VII. Concluding Remarks

The results of this study indicate the somewhat surprising result that optimal time-fixed direct-ascent interception from the surface of a planet or moon usually requires only a single impulse. The exceptions to this are those combinations of target initial lead angles β and sufficiently short transfer times for which the planet surface constraint is violated. In these cases, a two-impulse trajectory is required for the vehicle to remain outside the planet. The first impulse establishes a circular orbit at the planet surface; the second impulse transfers the vehicle from the surface to the target. The critical transfer time, at which any smaller time requires a constrained, two-impulse trajectory, can in some cases be significantly large, as shown by the geosynchronous target numerical example.

The reasons that more than one impulse typically is not required for optimal interception are 1) in contrast to rendezvous, both posigrade and retrograde trajectories can be optimal solutions because the final velocity is unconstrained, and 2) the period of the motion of the target relative to the launch point is relatively short because the launch point is at rest on the surface rather than in orbit about the planet. For a nonrotating planet, the period of the target-launch point relative motion is simply the target-orbit period. For this reason, the target-launch point geometry changes fairly rapidly during an initial coast period prior to the launch impulse, the main exception being a target orbit that is synchronous or near-synchronous with the planet rotation. The variety of intercept trajectories available utilizing these two aspects in combination for one-impulse trajectories is apparently great enough to include most optimal solutions.

The basic structure of the solution is that the transfer time for the time-open solution is finite and, for a nonrotating planet, is equal to the minimum-energy transfer time. For specified transfer times less than this value, the optimal solution is generally retrograde with an initial coast although, for a rotating planet, a posigrade trajectory is sometimes more economical because of the velocity of the launch point. For transfer times longer than the time-open value, a posigrade with initial coast transfer is typically optimal. For transfer times of the order of the synodic period of the target with respect to the launch point, the characteristics described earlier repeat periodically in all cases except the synchronous target orbit. In many cases, the optimal solution utilizes an initial coast, even when the specified transfer time is relatively small.

Appendix

As an illustrative simple example, the solution of optimal time-fixed and time-open impulsive interception in the field-free space is analyzed. This problem has also been treated by Edelbaum[12] and Marec.[13] The results obtained also illustrate the utility of time-fixed optimal solutions because, for a large class of field-free interceptions, the time-open optimal solution requires infinite time.

Consider the interceptor vehicle to be initially at rest at the origin ($\boldsymbol{r}_0 = \boldsymbol{v}_0 = \boldsymbol{0}$). The target has radius $\boldsymbol{r}_0^\#$ and velocity $\boldsymbol{v}_0^\#$ at the initial $t_0 = 0$. The trajectory of the target is then given by

$$\boldsymbol{r}^\#(t) = \boldsymbol{r}_0^\# + t\boldsymbol{v}_0^\# \tag{A1}$$

The primer vector differential equation (1) for both zero and uniform gravity is simply $\ddot{\boldsymbol{p}} = \boldsymbol{0}$, which has the solution $\boldsymbol{p}(t) = \boldsymbol{a} + t\boldsymbol{b}$. Applying the interception transversality condition $\boldsymbol{p}(t_f) = \boldsymbol{0}$ yields the conditions

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{b} = -\boldsymbol{a}/t_f \tag{A2}$$

$$\boldsymbol{p}(t) = \boldsymbol{a}(1 - t/t_f) \tag{A3}$$

Thus, the primer vector direction is constant and is opposite to the (constant) primer vector derivative $\boldsymbol{b}$. The primer vector magnitude decreases linearly from its initial value a to its final value zero.

Invoking the necessary condition that $p \leq 1$ with $p = 1$ at an impulse, it is evident that on an optimal solution $a \leq 1$, with $a < 1$ representing a no-impulse (all-coast) solution. This no-impulse solution applies only in the special case where $t_f\boldsymbol{v}_0^\# = -\boldsymbol{r}_0^\#$, i.e., the target is heading toward the launch point and will intercept it at time t_f!

The remaining admissible condition is $a = 1$, which implies a one-impulse solution with the impulse occurring at time $t_0 = 0$ (no initial coast period). That an initial coast is nonoptimal in the general case simplifies the analysis and the results.

Because the interceptor vehicle is initially at rest at the origin, its trajectory for $t > 0$ is simply $\boldsymbol{r}(t) = t\Delta\boldsymbol{v}_0$, where $\Delta\boldsymbol{v}_0$ is the velocity change due to the thrust impulse. (In contrast to the direct-ascent problem treated in the main text of this paper, the interceptor vehicle here is "in orbit" prior to the impulse because the rest state for $t < 0$ satisfies the field-free orbital equations of motion.) The condition for intercept is $\boldsymbol{r}(t_f) = \boldsymbol{r}^\#(t_f)$ which, combined with Eq. (A1), yields the required velocity change

$$\Delta\boldsymbol{v}_0 = \boldsymbol{v}_0^\# + \boldsymbol{r}_0^\#/t_f \tag{A4}$$

For the time-fixed case, Eq. (A4) describes the required velocity change for a given initial target state. The only optimization that has been performed is to disallow an initial coast prior to the first impulse. For the time-open solution,

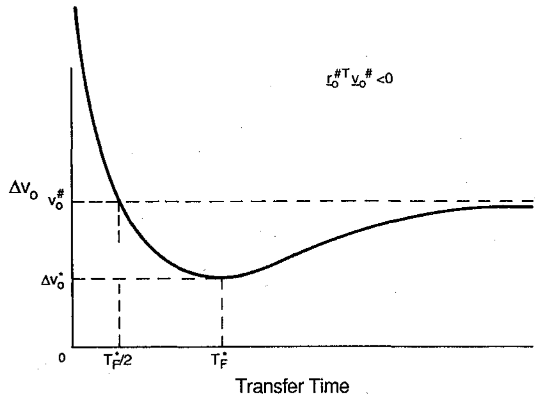

Fig. A1 Optimal ΔV vs transfer time, field-free space case 1.

however, another condition must be satisfied to determine the best value of the transfer time. As discussed in Sec. V, the primer vector derivative must be orthogonal to the relative velocity of the vehicle with respect to the target at the final time which, for the field-free case, is

$$\boldsymbol{v}_f - \boldsymbol{v}_f^{\#} = \Delta \boldsymbol{v}_0 - \boldsymbol{v}_0^{\#} = \boldsymbol{r}_0^{\#}/t_f \tag{A5}$$

where Eq. (A4) has been used. Because the primer derivative is $\boldsymbol{b} = -\boldsymbol{a}/t_f$ and $\boldsymbol{a}$ is the initial primer vector $\Delta \boldsymbol{v}_0/\Delta v_0$, the orthogonality of the primer derivative and the relative velocity of Eq. (A5) yields

$$\Delta \boldsymbol{v}_0^T \boldsymbol{r}_m^{\#} 0/t_f = (\boldsymbol{v}_0^{\#} + \boldsymbol{r}_0^{\#}/t_f)^T \boldsymbol{r}_0^{\#}/t_f = 0 \tag{A6}$$

Two separate cases emerge: 1) $\boldsymbol{r}_0^{\#T} \boldsymbol{v}_0^{\#} < 0$ and 2) $\boldsymbol{r}_0^{T\#} \boldsymbol{v}_0^{\#} \geq 0$. In case 1, the range to the target is decreasing and Eq. (A6) is satisfied for

$$t_f^* = -r_0^{\#2}/\boldsymbol{r}_0^{\#T} \boldsymbol{v}_0^{\#} > 0 \tag{A7}$$

In this case, the time-open optimal $\Delta \boldsymbol{v}_0$ is directed perpendicular to the initial target position $\boldsymbol{r}_0^{\#}$ vector and is equal to by Eq. (A7):

$$\Delta \boldsymbol{v}_0^* = \boldsymbol{h}^{\#} \times \boldsymbol{r}_0^{\#}/r_0^{\#2} \tag{A8}$$

with magnitude

$$\Delta v_0^* = h^{\#}/r_0^{\#} \tag{A9}$$

where $\boldsymbol{h}^{\#}$ is the constant angular momentum of the target, equal to $\boldsymbol{r}_0^{\#} \times \boldsymbol{v}_0^{\#}$. A plot of time-fixed optimal Δv_0 vs t_f for typical case 1 is shown in Fig. A1.

In case 2, the range to the target is increasing ($\boldsymbol{r}_0^{\#T} \boldsymbol{v}_0 \geq 0$), and there is no finite value of t_f that will satisfy Eq. (A6). In this case, an infimum of Δv_0 exists that is the limiting value as $t_f \to \infty$, given by Eq. (A4):

$$\Delta \boldsymbol{v}_0^* = \boldsymbol{v}_0^{\#} \tag{A10}$$

i.e., the vehicle is imparted a velocity equal to the target velocity. This time-open solution arbitrarily can be approximated closely by the finite-time solution

$$\Delta \boldsymbol{v}_0 = \boldsymbol{v}_0^{\#} + \boldsymbol{\varepsilon} \tag{A11}$$

where $\boldsymbol{\varepsilon}$ is given by (A4):

$$\boldsymbol{\varepsilon} = \boldsymbol{r}_0^{\#}/t_f \tag{A12}$$

As we make the value of t_f arbitrarily large, the magnitude of $\boldsymbol{\varepsilon}$ becomes arbitrarily small, approaching the time-open solution.

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